



$$R_d(Q) = \bigsqcup_{\text{finite}} G_d \cdot M = \overline{G_d \cdot \tilde{M}_d}$$

$\tilde{M}_d \in G_d \cdot \tilde{M}_d$  is characterized by

$$\text{Ext}^1(\tilde{M}_d, \tilde{M}_d) = 0$$

i.e.  $\tilde{M}_d$  is rigid.

For Dynkin quivers:

$M$  generic  $\stackrel{\text{def}}{\iff}$  its orbit is dense  $\iff M$  is rigid.

Degeneration order:

$M_1, M_2 \in R_d(Q)$  :

$M_1 \leq_{\text{deg}} M_2 \stackrel{\text{def}}{\iff} \overline{G_d \cdot M_1} \supseteq G_d \cdot M_2$

$\stackrel{\text{Thm}}{\iff} \dim \text{Hom}_Q(M_1, X) \leq \dim \text{Hom}_Q(M_2, X)$   
 $\forall X \in \text{Rep}(Q)$

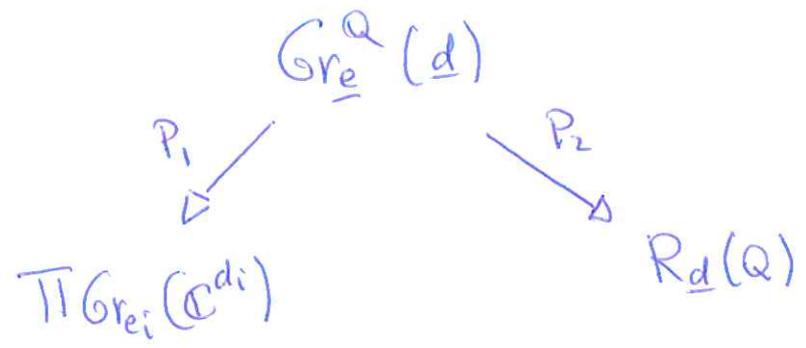
$\iff \dim \text{Hom}_Q(X, M_1) \leq \dim \text{Hom}_Q(X, M_2)$   
 $\forall X \in \text{Rep}(Q)$ .

Given  $\underline{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$  s.t.  $\underline{d} - \underline{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$  define

$$\text{Gr}_{\underline{e}}^Q(\underline{d}) = \left\{ \left( (U_i)_{i \in Q_0}, (f_{\alpha}) \in \prod \text{Gr}_{e_i}(\mathbb{C}^{d_i}) \times \underline{R}_d(\mathbb{Q}) \right) \right. \\ \left. f_{\alpha}(U_i) \subseteq U_j \quad \forall \alpha: i \rightarrow j \right\}$$

the "universal quiver grassmannians".

We have



Prop. (Schofield):

$P_1$  is a vector bundle of rank  $\sum_{i \rightarrow j} d_i d_j + e_i e_j - e_i d_j$

Cor:  $\text{Gr}_{\underline{e}}^Q(\underline{d})$  is smooth and irreducible and

$$\dim \text{Gr}_{\underline{e}}^Q(\underline{d}) = \langle \underline{e}, \underline{d} - \underline{e} \rangle_{\mathbb{Q}} + \dim \underline{R}_d(\mathbb{Q})$$

$$\langle \underline{x}, \underline{y} \rangle_{\mathbb{Q}} := \sum_{i \in Q_0} x_i y_i - \sum_{i \rightarrow j} x_i y_j$$

$P_2$  is proper and  $G_d$ -equivariant.

Def:  $Gr_e(M) := P_2^{-1}(M)$  is the quiver Grassmannian of  $e$ -dimensional subrepresentations of  $M$ .

$$\text{Im } P_2 \stackrel{\text{closed}}{\subseteq} R_d(Q)$$

$$\text{Im } P_2 = \{ M \mid \exists U \subseteq_Q M \text{ } \underline{\dim} U = e \} = ?$$

Thm [Reineke-Kerkman '15]:

$$M \in \text{Im } P_2 \iff \dim \text{Hom}_Q(X, M) \geq \langle \underline{\dim} X, e \rangle \\ \forall X \text{ indec. reps. of } Q.$$

RK:  $\text{Im } P_2 = R_d(Q) \iff \tilde{M}_d \in \text{Im } P_2.$

Prop. [Schofield, Bongartz, CI-Fujita-Reineke, CI]:

$$Gr_e(\tilde{M}_d) \neq \emptyset \iff \text{Ext}^1(\tilde{M}_e, \tilde{M}_{d-e}) = 0$$

In this case, there is s.e.s.

$$0 \rightarrow \tilde{M}_e \rightarrow \tilde{M}_d \rightarrow \tilde{M}_{d-e} \rightarrow 0.$$

Moreover,  $Gr_e(\tilde{M}_d)$  is smooth and irreducible of dimension  $\langle e, d-e \rangle$ .

RK: If  $M \in \text{Im } P_2$  then

$$\dim I \geq \langle \underline{e}, \underline{d} - \underline{e} \rangle = \dim \text{Gr}_{\underline{e}}^{\underline{Q}}(\underline{d}) - \dim R_{\underline{d}}(\underline{Q})$$

for every irreducible component of  $\text{Gr}_{\underline{e}}(M) = P_2^{-1}(M)$ .

Def: We say that a quiver Grassmannian  $\text{Gr}_{\underline{e}}(M)$  has minimal dimension if

$$\dim \text{Gr}_{\underline{e}}(M) = \langle \underline{e}, \underline{d} - \underline{e} \rangle.$$

RK:  $\langle \underline{e}, \underline{d} - \underline{e} \rangle$  is not always a possible dimension.

Aim: Study the geometry of quiver Grassmannians of Dynkin Type.

Thm [Bongartz]: Let  $M_1, M_2 \in R_{\underline{d}}(\underline{Q})$  s.t.  $M_1 \leq_{\text{deg}} M_2$ .

Let  $X \subseteq M_2$ . If

$$\dim \text{Hom}_{\underline{Q}}(X, M_2) = \dim \text{Hom}_{\underline{Q}}(X, M_1)$$

then  $X \subseteq M_1$  too. In this case, every quotient of  $M_2$  by  $X$  is a degeneration of the generic quotient of  $M_1$  by  $X$ .

Smooth means

$$\forall p \in \text{Gr}_e(M) \quad \dim T_p(\text{Gr}_e(M)) = \dim \text{Gr}_e(M).$$

How to compute  $\dim T_p(\text{Gr}_e(M))$ ?

Prop. [Schofield, Caldero-Reineke]:

Let  $N \in \text{Gr}_e(M)$ . Then

$$T_N(\text{Gr}_e(M)) \simeq \text{Hom}_Q(N, M/i(N))$$

$\downarrow$   
 $N \subset M$

So  $\text{Gr}_e(M)$  is smooth of minimal dimension if and only if

$$\dim \text{Hom}_Q(N, M/N) = \langle e, d-e \rangle \quad \forall N \in \text{Gr}_e(M)$$

$$\Leftrightarrow \text{Ext}_Q^1(N, M/N) = \{0\} \quad \forall N \in \text{Gr}_e(M).$$

Can we describe those varieties?

Put:

$$\mathcal{J}_e(d) = \{ M \in \text{Im } \mathbb{P}_2 \subset \mathbb{R}_d \mid \text{Gr}_e(M) \text{ is smooth of minimal dimension} \}$$

Prop(CI, '16)

$$\mathcal{L}_{\underline{e}}(\underline{d}) \neq \emptyset \iff \tilde{M}_{\underline{d}} \in \text{Imp}_2 \iff \text{Ext}^1(\tilde{M}_{\underline{e}}, \tilde{M}_{\underline{d}-\underline{e}}) = \{0\}.$$

Recall:

Two algebraic varieties  $X_1$  and  $X_2$  are called

deformation equivalent if  $\exists$

smooth proper family  $f: \mathcal{X} \rightarrow \mathcal{Y}$ ,

$\mathcal{Y}$  connected, and  $p_1, p_2 \in \mathcal{Y}$  s.t.

$$f^{-1}(p_1) = X_1, \quad f^{-1}(p_2) = X_2.$$

In this case  $X_1$  and  $X_2$  are diffeomorphic

(Ehresmann) and have the same Hodge numbers.

In particular,  $P_{X_1}(q) = P_{X_2}(q)$  and  $\chi(X_1) = \chi(X_2)$ .

Thm 1 [CI, '16]: ~~Let~~ Let  $\underline{e}, \underline{d} \in \mathbb{Z}_{\geq 0}^{\mathbb{Q}_0}$  s.t.  $\mathcal{L}_{\underline{e}}(\underline{d}) \neq \emptyset$ .

Let  $M_1, M_2 \in \mathcal{L}_{\underline{e}}(\underline{d})$ . Then

$G_{\underline{e}}(M_1)$  and  $G_{\underline{e}}(M_2)$  are

deformation-equivalent. In particular,

$$P_{G_{\underline{e}}(M_1)}(q) = P_{G_{\underline{e}}(M_2)}(q), \quad \chi(G_{\underline{e}}(M_1)) = \chi(G_{\underline{e}}(M_2)).$$

Thm [CI, '16]:

$M_1, M_2 \in \underline{S}_e(d) \implies \text{Gr}_e(M_1)$  and  $\text{Gr}_e(M_2)$  are d.e.

Proof:

$$\bar{p} = p_2| : p_2^{-1}(\underline{S}_e(d)) \longrightarrow \underline{S}_e(d)$$

is a proper map (being the restriction of a proper map).

$\underline{S}_e(d)$  is open and dense in  $\mathbb{R}_d(\mathbb{Q})$

(since contains  $\tilde{M}_d$ )

$\implies p_2^{-1}(\underline{S}_e(d))$  is open and dense in  $\text{Gr}_e^{\mathbb{Q}}(d)$ .

$\implies \bar{p}$  is a proper map between smooth and irreducible varieties.

All fibers of  $\bar{p}$  have the same dimension

$\implies \bar{p}$  is flat  
Matsumura

All fibers of  $\bar{p}$  are smooth +  $\bar{p}$  flat

$\implies \bar{p}$  is smooth.  
Ehresmann  $\square$

Thus, there is only one Topological space  $X_e(d)$

~~and~~ all attached to  $S_e(d)$  so that

given  $M \in S_e(d)$   $G_e(d)$  is  $X_e(d)$  endowed with some complex structure.

Applications:

CC-formula:

Def: ) Given  $M \in R_d(Q)$ ,

$$F_M(y_1, \dots, y_n) = \sum_e \chi(G_e(M)) y^e \in \mathbb{Z}[y_1, \dots, y_n]$$

)  $0 \rightarrow M \rightarrow I_0(M) \rightarrow I_1(M) \rightarrow 0$  min. inj. res. of  $M$

$$\underline{g}_M := \text{ind}_M = [I_0(M)] - [I_1(M)] \in K_0(Q) \cong \mathbb{Z}^n$$

$[s_i] \mapsto e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

)  $B_Q = (b_{ij})_{i,j \in Q_0} \in \text{Mat}_{n \times n}(\mathbb{Z})$

$$b_{ij} = \# \{ j \rightarrow i \text{ arrow in } Q \} - \# \{ i \rightarrow j \text{ arrow in } Q \}.$$

$$) \quad CC_M(x_1, \dots, x_n) = F_M(\hat{y}_1, \dots, \hat{y}_m) \underline{x}^{\underline{g}_M}$$

$$= \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}}(M)) \underline{x}^{\text{Be} + \underline{g}_M} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$\hat{y}_i := \underline{x}^{\text{Be}_i}$$

Thm (CC, '06):

$$\{CC_M : M \text{ indec.}\} \cup \{x_1, \dots, x_n\}$$

is the set of all cluster variables of the (coefficient-free)  
cluster algebra  $A_Q$  associated with  $Q$ , i.e.  
with initial seed  $(B_Q, \{x_1, \dots, x_n\})$ .

Key Lemmas:

Recall: An almost split sequence in  $\text{Rep}(Q)$  is a s.e.s.

$$\zeta: 0 \rightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \rightarrow 0$$

s.t.

1)  $L, N$  are indec.

2)  $\zeta$  does not split

3)  $\forall X \xrightarrow{s} N$  which is not split epi

$$\exists t: X \rightarrow M \text{ s.t. } s = \pi t.$$

For every indecomposable representation  $M$  of  $Q$  which is not projective,  $\exists$  (a unique) almost split sequence

$$0 \rightarrow CM \xrightarrow{i} E \xrightarrow{\pi} M \rightarrow 0$$

Key Lemma (CC '06):

$$\boxed{\chi(\text{Gr}_{\underline{e}}(E)) = \chi(\text{Gr}_{\underline{e}}(M \oplus CM)) \quad \forall \underline{e} \neq \underline{\dim} M} \quad (*)$$

$$\chi(\text{Gr}_{\underline{\dim} M}(M)) = 1$$

$$\chi(\text{Gr}_{\underline{\dim} M}(E)) = 0$$

Thm 2 [CI '46]:  $\text{Gr}_{\underline{e}}(E)$  and  $\text{Gr}_{\underline{e}}(M \oplus CM)$  are deformation equivalent,  $\forall \underline{e} \neq \underline{\dim} M$ .

In particular (\*) holds. Moreover

proof: It is easy to prove that  $E = \tilde{M}_d$  is rigid.

So it's enough to check that

$$\text{Ext}^1(N, \frac{M \oplus CM}{N}) = 0 \quad \forall N \in \text{Gr}_{\underline{e}}(M \oplus CM).$$

This follows immediately from 1) 2) and 3).

□

RK: In type A they are isomorphic! (Butler-Ringel)

In type D and E this is not the case

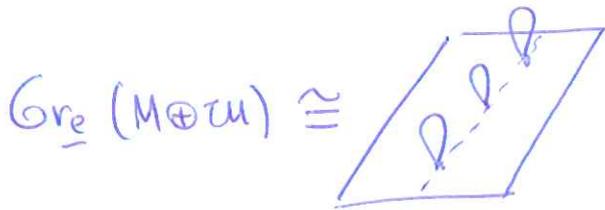
Ex:  $Q: \begin{array}{ccc} 1 & 2 & 3 \\ & \downarrow & \downarrow \\ & & 4 \end{array}$

$$0 \rightarrow \begin{array}{c} 111 \\ 2 \end{array} \rightarrow \underbrace{\begin{array}{c} 110 \\ 1 \end{array} \oplus \begin{array}{c} 101 \\ 1 \end{array} \oplus \begin{array}{c} 011 \\ 1 \end{array}}_E \rightarrow \begin{array}{c} 111 \\ 1 \end{array} \rightarrow 0$$

$\underline{e} := \dim \mathcal{M} = \begin{array}{c} 111 \\ 2 \end{array}$



$\mathbb{P}^2$  blown up in three generic points ( $\simeq$  Fano)



$\mathbb{P}^2$  blown-up in three collinear points ( $\simeq$  NOT Fano)

## Positivity and homology

Thm [CI]: Suppose  $\mathcal{P}_{e,d} \neq \emptyset$  and  $M \in \mathcal{P}_{e,d}$ . Then

a)  $\text{Gr}_e(M)$  has no odd cohomology. In particular

$$\chi(\text{Gr}_e(M)) \geq 0$$

b)  $\text{Hodd}(\text{Gr}_e(M))$  is zero

c)  $\text{Heven}(\text{Gr}_e(M))$  is torsion-free

RK: The proof is by induction on the AR-quiver of  $Q$ , and use Thm 2.

.) a) was proved by Caldero and Keller, Nakajim, F. Qin...

.) In the process of the proof I found the formula

$$P_{\text{Gr}_e(E)}(q) = \sum_{f+g=e} q^{2\langle f, \text{dim} M - g \rangle} P_{\text{Gr}_f(M)}(q) P_{\text{Gr}_g(M)}(q)$$

which follows immediately from Thm 2.

With F. Qin we verified that this formula provides another proof for the quantum version of CC.